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On Witten-type deformation of osp(1/2) Algebra

W.Chung

Theory Group, Department of Physics, College of Natural Sciences,
Gyeongsang National University,
Jinju, 660-701, Korea

Abstract

In this paper Witten type deformation of osp(1/2) algbera is introduced and its realization and matrix representation are obtained. The matrix representation is shown to be possible only when the dimension is odd.

1 Introduction

Quantum groups or q-deformation of Lie algebra implies some specific deformation of classical Lie algebra. From a mathematical point of view, it is a non-commutative associative quasi-triangular Hopf algebra. The first quantum deformations of the classical Lie algebra $su(N)$ treated the elements in the Cartan subalgebra on a different footing [1-4]. For $su(2)$, the deformed algebra is defined as

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2H], \tag{1}$$

where the q-number is defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{2}$$

This algebra is sometimes called the transcendental deformation because the q-number is given in terms of the transcendental functions. It is known by Woronowicz [5] and Witten [6] that another type of deformation of ordinary $su(2)$ algebra is possible . They treated the generators on a similar footing. And their work enabled Curtright and Zachos [7] and Fairlie [8] to find the explicit invertible functionals that map su(2) algebra generators to those of q-deformed algebra . These functionals are shown to deform su(2) continuously and reversibly (except for the special value of deformation parameter) into each of the quantum universal enveloping algebraic structures. They also applied their method to an obvious two parameter extension which covers both Woronowicz's and Witten's form, which is given by

$$rHJ_+ - r^{-1}J_+H = J_+,$$

$$rJ_-H - r^{-1}HJ_- = J_-,$$
$$s^{-1}J_+J_- - sJ_-J_+ = H. \tag{3}$$

In this algebra $s = r^2$ reproduces the Witten's algebra while $s = \sqrt{r}$ reproduces the Woronowicz's algebra .

In this paper we construct the Witten type deformation for the $osp(1/2)$ algebra and obtain the difference operator realization and matrix representation for this algebra . The triangular deformation of superalgebra is well-known in many papers [9-12].

2 Witten Type Deformation of OSP(1/2) Algebra

In this section we discuss the Witten type deformation of $osp(1/2)$ algebra. Let us start with the following form of the q-deformed catesian $osp(1/2)$ algebra;

$$\{V_-, V_+\}_q = H,$$
$$\{V_-, V_-\}_q = J_-,$$
$$\{V_+, V_+\}_q = J_+,$$
$$[H, V_+]_q = V_+,$$
$$[V_-, H]_q = V_-, \tag{4}$$

where the qummutator and antiqummutator are defined as

$$[A, B]_q = AB - qBA, \quad \{A, B\}_q = AB + qBA. \tag{5}$$

From the eq.(4), especially the second and third equations, one can obtain the remaining commutation relations of the Witten type deformation of $osp(1/2)$ algebra which is given by

$$[J_\pm, V_\pm] = 0,$$
$$[V_-, J_+]_{q^2} = [2]V_+,$$
$$[J_-, V_+]_{q^2} = [2]V_-,$$
$$[H, J_+]_{q^2} = [2]J_+,$$
$$[J_-, H]_{q^2} = [2]J_-,$$
$$[J_-, J_+]_{q^4} = [2]^2(qH + (1-q)V_-V_+), \tag{6}$$

where the q-number $[x]$ is defined as $[x] = \frac{q^x - 1}{q-1}$.

If we replace the three bosonic generators by

$$H \to q^{-1}[2]H, \quad J_\pm \to \pm q^{-1}[2]^{3/2}J_\pm$$

and decouple the fermionic generators in order to compare this algbera with its bosonic analogue (3), we have the following algebra

$$q^{-1}HJ_+ - qJ_+H = J_+,$$

$$q^{-1}J_-H - qHJ_- = J_-,$$

$$q^2J_+J_- - q^{-2}J_-J_+ = H, \tag{7}$$

This correspond to the case that $r = s^2$ in eq.(3), which implies that this algebra is supersymmetric extension of Witten's algebra.

A glance at the last equation of eq.(6) indicates that the qummutator of the two even generators (J_+, J_-) can not be written in terms of the function in H only . However , when the deformation parameter q goes over to 1, the left hand side of the

last equation of eq.(2) becomes an ordinary commutator , which is given in terms of the function in H only.

From algebra(4) we can set

$$V_+V_- = F(H), \tag{8}$$

where $F(H)$ is a function in H which will be fixed later. Let f be an inverse function of F; then we get

$$H = f(V_+V_-) = \Sigma_n c_n (V_+V_-)^n. \tag{9}$$

From the 4th and 5th relation of (4) we have

$$[H, V_+]_q$$

$$= [\Sigma_n c_n (V_+V_-)^n, V_+]_q$$

$$= \Sigma_n c_n [V_+(V_+V_-)^n - qV_+(V_+V_-)^n]$$

$$= V_+[f(V_-V_+) - qf(V_+V_-)] = V_+. \tag{10}$$

Thus we have

$$f(V_-V_+) = qf(V_+V_-) + 1,$$

which implies

$$V_-V_+ = F(qH + 1). \tag{11}$$

Inserting (8) and (10) into the 1st relation of (4) gives

$$F(1 + qH) + qF(H) = H. \tag{12}$$

We can easily obtain the Casimir operator for the algebra (4) as follows;

$$C = V_+V_- - \frac{1}{2q}(H - \frac{1}{1+q}). \tag{13}$$

Let us denote by $|c,m>$ a simultaneous eigenvector of the commuting hermitian operator H and C, where we have

$$H|c,m>=m|c,m>, \quad C|c,m>=c|c,m> . \tag{14}$$

From the algebra (4) and the fact that V_- is a hermitian conjugate of V_+, we see that

$$V_+|c,m>=a(c,qm+1)|c,qm+1>, \tag{15}$$

$$V_-|c,m>=a(c,m)|c,q^{-1}m-q^{-1}> . \tag{16}$$

Using the Casimir operator and algebra (4) we have

$$V_-|c,m>=\sqrt{\frac{j+m}{2q}}|c,q^{-1}(m-1)>, \tag{17}$$

$$V_+|c,m>=\sqrt{\frac{j+qm+1}{2q}}|c,qm+1)>, \tag{18}$$

where we set $m\geq -j$. If we define

$$|j,n>=|c,-q^n j+[n]>, \quad [n]=\frac{1-q^n}{1-q}$$

then we have a simpler representation of algebra (4) as follows;

$$H|j,n>=([n]-q^n j)|j,n>, \quad (n=0,1,2,\cdots)$$

$$V_-|j,n>=\sqrt{\frac{1+(1-q)j}{2q}}\sqrt{[n]}|j,n-1>,$$

$$V_+|j,n>=\sqrt{\frac{1+(1-q)j}{2q}}\sqrt{[n+1]}|j,n+1> . \tag{19}$$

So the representation is infinite dimensional and is bouned from below.

3 Realization

In this section we discuss the realization of Witten type deformation of $osp(1/2)$ algebra. In doing so we need the new type of the deformed oscillator algebra. Now let us consider the following deformed oscillator algebra

$$[a,a^+]_q=1, \quad [N,a^+]_q=a^+, \quad [a,N]_q=a \tag{20}$$

where a is assumed to be a hermitian conjugate of a^+ and the number operator N is assumed to be hermitian. Then the third relation of eq.(20) is not necessary any more because it is obtained by taking the complex conjugate of the second relation of eq.(20). The first relation itself takes self-hermitian form. If we replace the

qummutators with the ordinary commutators in the second and the third relation of eq.(20), this algebra reduces to the standard q-deformed boson algebra. From the definition (20), we have

$$a^+a = N, \quad aa^+ = 1 + qN. \tag{21}$$

If we assume that there exists a unique vacuum state $|0>$ satisfying

$$a|0>= 0, \tag{22}$$

we have, from the relation (21),

$$N|0>= 0. \tag{23}$$

Then the higher states can be obtained by acting the creation operator a^+ on the vacuum state successively. Let the state $|n>$ be proportional to $(a^+)^n|0>$. Then we have

$$N|n>= [n]|n>, \quad n = 0, 1, 2, \cdots. \tag{24}$$

So the eigenvalues of the number operator N become a sequence of the q-numbers. Using the second and third relation of eq.(20) , we get

$$a|n>= \sqrt{[n]}|n-1>, \quad a^+|n>= \sqrt{[n+1]}|n+1> . \tag{25}$$

From this new deformed boson algebra we can obtain the realization for the $osp(1/2)$ algebra as follows;

$$v_- = \frac{1}{\sqrt{1+q}}a$$

$$v_+ = \frac{1}{\sqrt{1+q}}a^+$$

$$H = \frac{1}{1+q}(aa^+ + qa^+a). \tag{26}$$

Now we can obtain the difference operator realization for the $osp(1/2)$ algebra. The coordinate realization of algebra (20) is given by

$$a^+ = x, \quad a = D, \quad N = xD, \tag{27}$$

where the q-derivative D is defined as

$$Df(x) = \frac{f(qx) - f(x)}{x(q-1)} \tag{28}$$

Using this relation, we can write the difference operator realization for the Witten type deformation of $osp(1/2)$ algebra as follows;

$$v_- = \frac{1}{\sqrt{1+q}}D, \quad v_+ = \frac{1}{\sqrt{1+q}}x, \quad H = \frac{1}{1+q}(1 + 2qxD) \tag{29}$$

4 Matrix Representation

Now we will obtain the matrix representation of the algebra (4) by assuming that H is diagonal and $V_+(V_-)$ are super (sub) diagonal;

$$(H)_{ij} = h_i \delta_{ij},$$

$$\begin{aligned} (V_+)_{ij} &= v_i \delta_{i+1,j}, \\ (V_-)_{ij} &= v_j \delta_{i,j+1}, \end{aligned} \tag{30}$$

where the last relation is obtained from

the second relation by using the fact that V_- is hermitian conjugate

to V_+. Then the forth relation of eq.(4) determines h_i through the recurrence relation;

$$h_i - q h_{i+1} = 1, \quad (i = 1, 2, \cdots, n-1). \tag{31}$$

The relation (31) is easily solved and the solution is given by

$$h_i = q^{n-i} h_n + \frac{q^{-1}(q^{n-i} - 1)}{1 - q^{-1}}, \tag{32}$$

where h_n will be determined by using the another relations of eq.(4). Using the first relation of eq.(4), we get

$$v_{i-1}^2 + q v_i^2 = h_i, \quad (i = 1, 2, \cdots, n); \quad v_0 = 0 \; and \; v_n = 0. \tag{33}$$

Using the eq.(32) and checking the consistency of eq.(33), we obtain the following identity;

$$\Sigma_{k=1}^{n} (-q^{-1})^{n-k} h_k = 0. \tag{34}$$

This relation and eq.(32) fix the concrete form of the value of h_n. Inseting eq.(32) into the eq.(34) produces

$$h_n \frac{(-)^n - 1}{2} = \frac{1}{1-q} \frac{(-)^n - 1}{2} + \frac{1}{1-q} \frac{q + (-q^{-1})^{n-1}}{1+q}. \tag{35}$$

For the case that n is even, the eq.(35) does not give the solution for h_n; so we assume that n is odd. Then we have

$$h_n = \frac{1}{1-q} + \frac{q + q^{-n+1}}{(q-1)(q+1)}. \tag{36}$$

Inserting eq.(36) into eq.(32) we obtain

$$h_i = \frac{1}{1-q} \{ 1 - \frac{q}{1+q} (q^n + 1) q^{-i} \}. \tag{37}$$

Using eq.(33) we can perform the partial summations and obtain a closed expression for v_i given by;

$$v_i = \sqrt{\frac{1}{1-q^2}[1-(-q^{-1})^i + \frac{(-)^i - 1}{2}(q^n+1)q^{-i}]}. \tag{38}$$

For $n = 3$ case we can obtain the explicit matrix representation as follows;

$$H = \begin{pmatrix} q & 0 & 0 \\ 0 & 1-q^{-1} & 0 \\ 0 & 0 & -q^{-2} \end{pmatrix}$$

$$V_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & iq^{-1} \\ 0 & 0 & 0 \end{pmatrix}$$

$$V_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -iq^{-1} & 0 \end{pmatrix}, \tag{39}$$

where we assumed that q is real.

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